Math 210A Lecture 18 Notes

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1 Composition Series

1.1 Restrictions on simple groups

Lemma 1.1. *Let P, Q be Sylow p-subgroups of a group G. $P \cap Q = P \cap N_G(Q)$.*

Proof. Let $H = P \cap N_G(Q)$. We know that $H \leq N_G(Q)$, so $HQ = QH$. So $HQ \leq G$. Since $|HQ| = |H||Q|/|H \cap Q|$, HQ is a p-group. So $H \leq Q$ since Q is a Sylow p-subgroup. $\square$

Proposition 1.1. *Let G be a finite group and let $P^n \mid\mid |G|$ for $n \geq 1$. Assume that for all Sylow p subgroups $P \neq Q$, $|P \cap Q| \leq p^{n-r}$. Then $n_p(G) = 1 \pmod{p}^r$.*

Proof. $P \circlearrowright \mathrm{Syl}_p(G)$ by conjugation. Note that $p^n \mid [P : P \cap Q] = [P : P_Q] = |\text{orbit of } Q|$. We can count

$$n_p(G) = \sum_{\text{orbits}} |\text{orbit}| \equiv 1 \pmod{p^r}.$$

$\square$

Proposition 1.2. *Every simple group of order 60 is isomorphic to A_5.*

Proof. Factor $60 = 4 \cdot 3 \cdot 5$. Then $n_5(G) = 6$, $n_3(G) = 4$ or 10, and $n_2(G) = 3, 5$ or 15. We cannot have $n_3(G) = 4$ or $n_2(G) = 3$. If $n_2(G) = 5$, then G is isomorphic to a subgroup of $S_5 \cong S_{\mathrm{Syl}_2(G)}$. So the image of G has index 2. If $G \neq A_5$, then $G \cap A_5$ has index 2 in A_5. Since subgroups of index 2 are normal, we get $G \cap A_5 \trianglelefteq A_5$, contradicting the fact that A_5 is simple. So in this case, $G \cong A_5$.

If $n_2(G) = 15$, then $15 \not\equiv 2 \pmod 4$, so we have $P, Q \in \mathrm{Syl}_2(G)$ with $|P \cap Q| = 2$. Then $N_G(P \cap Q) \supseteq PQ$. So $|N_G(P \cap Q)| > 4$ and is a multiple of 4 dividing 60. So $|N_G(P \cap Q)| \in \{12, 20, 60\}$. If $|P \vee Q| = 60$, then $N_G(P \cap Q) = G$, so $P \cap Q \trianglelefteq G$. If $|M| = 12$ or 20, then G acts on G/M, of order ≤ 5. So G is isomorphic to a subgroup of S_3 or S_5. S_3 is impossible because G is too large, and we have already treated the case of S_5. $\square$

Proposition 1.3. *There are no simple groups of order* $396 = 4 \cdot 9 \cdot 11$.

Proof. If G is simple, then $n_{11}(G) = 12 = [G : N_G(P)]$, where P is a Sylow 11-subgroup. Then $|N_G(P)| = 33$. So G is isomorphic to a subgroup of S_{12}, and we get $N_G(P) \leq N_{S_{12}}(P)$. Then P is still Sylow 11 in S_12, so $n_{11}(S_{12}) \mid 12!/33 = 10! \cdot 4$. We can count $n_{11}(S_{12}) = 12!/(11 \cdot 10) = 9! \cdot 12$. But $12 \nmid 40$, so we have a contradiction. □

1.2 Composition series

Definition 1.1. Let G be a group. A **series** is a collection $(H_i)_{i\in\mathbb{Z}}$ of subgroups of G such that $H_{i-1} \leq H_i$ for all i.

Definition 1.2. A series is **ascending** if $H_i = 1$ for all i sufficiently small. A series is **descending** if $H_i = G$ for all sufficiently large i. A series is **finite** if it is both ascending and descending.

In the descending case, we often take $H_i \leq H_{i-1}$ and only deal with $i \geq 0$. If the series is finite and we write

$$1 = H_0 \leq H_1 \leq \cdots \leq H_{t-1} \leq H_t = G$$

with $H_i \neq H_{i-1}$ for all i, then we say that t is the length of the series.

Definition 1.3. A finite series is **subnormal** if $H_{i-1} \trianglelefteq H_i$ for all i. A finite series is **normal** if $H_{i-1} \trianglelefteq G$ for all i.

Definition 1.4. A **composition series** is a subnormal series such that H_i/H_{i-1} are all simple or trivial. The H_i/H_{i-1} are called **composition factors**.

Example 1.1. In the composition series

$$1 \trianglelefteq A_5 \trianglelefteq S_5$$

the composition factors are S_5 and $\mathbb{Z}/2\mathbb{Z}$.

Example 1.2. In the composition series

$$1 \trianglelefteq p^{n-1}\mathbb{Z}/p^n\mathbb{Z} \trianglelefteq p^{n-2}/p^n\mathbb{Z} \trianglelefteq \cdots \trianglelefteq p\mathbb{Z}/p^n\mathbb{Z} \trianglelefteq \mathbb{Z}/p^n\mathbb{Z}$$

the composition factors are all $\mathbb{Z}/p\mathbb{Z}$.

Example 1.3. In the composition series

$$1 \trianglelefteq \mathbb{Z}/2\mathbb{Z} \trianglelefteq (\mathbb{Z}/2\mathbb{Z})^2 \trianglelefteq A_4 \trianglelefteq S_4$$

the composition factors are $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}$.

Lemma 1.2. *Given a composition series and $N \trianglelefteq G$,*

1. *We have a composition series* $H_{f(i)} \cap N$ *with* $f : \{0, \ldots, s\} \to \{0, \ldots, t\}$ *with* $f(0) = 0$ *with the* i*-th factor* $H_{f(i)}/H_{f(i)-1} \cong H_{f(i)}/H_{f(i-1)}$

2. $\overline{H_i} = H_i/(H_i \cap N)$*, and we have a composition series for* G/N *of the form* $\overline{H_{f(i)}}$ *with* $f' : \{0, \ldots, r\} \to \{0, \ldots t\}$ *increasing with* $f(0) = 0$ *and composition factors* $H_{f'(i)}/H_{f'(i)-1}$

3. $\mathrm{im}(f) \cup \mathrm{im}(f') = \{0, \ldots, t\}$*, and* $r + s = t$*.*